

A relation for a class of Racah polynomials

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Abstract

In this paper we derive a relation for a class of Racah polynomials that appear in a conjecture of Kresch and Tamvakis. The relation follows from an inversion formula for a transformation of a discrete sequence of complex numbers $\{x_n\}_{n=0}^{\infty}$. As a result of our inversion formula, we also obtain other combinatorial identities.

1 Introduction

Let p and q be non-negative integers. Let $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, z \in \mathbb{C}$. The hypergeometric series of type ${}_pF_q$ with numerator parameters a_1, a_2, \dots, a_p and denominator parameters b_1, b_2, \dots, b_q is defined by

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{n! (b_1)_n (b_2)_n \cdots (b_q)_n} z^n, \quad (1.1)$$

where the rising factorial $(a)_n$ is given by

$$(a)_n = \begin{cases} a(a+1) \cdots (a+n-1), & n > 0, \\ 1, & n = 0. \end{cases}$$

If no numerator parameter is a non-positive integer, we need no denominator parameter to be a non-positive integer. In this case, the series in (1.1) converges absolutely for all z if $p < q + 1$. If $p > q + 1$, the series converges only when $z = 0$. In the case $p = q + 1$, the series converges absolutely if $|z| < 1$ or if $|z| = 1$ and $\operatorname{Re}(\sum_{i=1}^q b_i - \sum_{i=1}^p a_i) > 0$ (see [1, p. 8]).

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If a numerator parameter is a non-positive integer, then, letting $-n$ be the largest non-positive integer numerator parameter, only the first $n+1$ terms of the series (1.1) are non-zero and the series is said to terminate. In this case, we require that no denominator parameter be in the set $\{-n+1, -n+2, \dots\}$. We note that (1.1) reduces to a polynomial in z of degree n .

When $z = 1$, we say that the series is of unit argument and of type ${}_pF_q(1)$. If $\sum_{i=1}^q b_i - \sum_{i=1}^p a_i = 1$, the series is called Saalschützian.

We will make use of the Chu-Vandermonde formula (see [1, p. 3]) for the sum of a terminating ${}_2F_1(1)$ series:

$${}_2F_1 \left[\begin{matrix} -n, a; \\ b; \end{matrix} 1 \right] = \frac{(b-a)_n}{(b)_n}. \quad (1.2)$$

We will also use the binomial coefficient identities

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \quad 1 \leq k \leq n, \quad (1.3)$$

$$\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}, \quad 0 \leq m \leq k \leq n, \quad (1.4)$$

and

$$\frac{\binom{m+k}{m}}{\binom{n+k}{n}} = \frac{\binom{n}{m}}{\binom{n+k}{m+k}}, \quad 0 \leq m \leq n, \quad k \geq 0. \quad (1.5)$$

The Racah polynomials, first given by Wilson [7], are defined by (see also [4])

$$\begin{aligned} & R_n(\lambda(x); \alpha, \beta, \gamma, \delta) \\ &= {}_4F_3 \left[\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1; \\ \alpha + 1, \beta + \delta + 1, \gamma + 1; \end{matrix} 1 \right], \\ & n = 0, 1, 2, \dots, N, \end{aligned} \quad (1.6)$$

where

$$\lambda(x) = x(x + \gamma + \delta + 1)$$

and

$\alpha+1 = -N$ or $\beta+\delta+1 = -N$ or $\gamma+1 = -N$, with N a non-negative integer.

We note that the Racah polynomials are terminating Saalschützian ${}_4F_3(1)$ hypergeometric series.

The special case $\alpha = \beta = \gamma + \delta = 0, \gamma = T$, where T is a positive integer leads to the definition of

$$R_n(s, T) := R_n(\lambda(s); 0, 0, T, -T) = {}_4F_3 \left[\begin{matrix} -n, n+1, -s, s+1 \\ 1, 1-T, 1+T \end{matrix}; 1 \right], \quad (1.7)$$

where $0 \leq n, s \leq T-1$.

It is conjectured by Kresch and Tamvakis [5] that

$$|R_n(s, T)| \leq 1 \quad (1.8)$$

for all $0 \leq n, s \leq T-1, T \geq 1$. Special cases of the conjecture are proven by Kresch and Tamvakis in [5]. Special cases of the conjecture are also proven by Ismail and Simeonov [2]. Furthermore, Ismail and Simeonov demonstrate asymptotics for $R_n(s, T)$ in [2] that are in agreement with the conjecture.

2 Main result

Definition 2.1. Let $\{x_n\}_{n=0}^\infty \subseteq \mathbb{C}$. For each $n \geq 0$, we define

$$\tilde{x}_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} x_k. \quad (2.1)$$

We note that the transformation (2.1) is linear.

In view of the formulas

$$(-1)^k \binom{n}{k} = \frac{(-n)_k}{k!} \quad (2.2)$$

and

$$\binom{n+k}{k} = \frac{(n+1)_k}{k!}, \quad (2.3)$$

equation (2.1) can also be written as

$$\tilde{x}_n = \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{k! k!} x_k. \quad (2.4)$$

We remark that the transformation in Definition 2.1 is inspired by the binomial transform (introduced by Knuth in [3]) of a sequence $\{x_n\}_{n=0}^\infty \subseteq \mathbb{C}$ defined by

$$\hat{x}_n = \sum_{k=0}^n (-1)^k \binom{n}{k} x_k, \quad n \geq 0. \quad (2.5)$$

The inversion formula for the binomial transform is well-known (see [6]) and is

$$x_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \hat{x}_k, \quad n \geq 0. \quad (2.6)$$

Certain terminating hypergeometric series can be considered as binomial transforms. For example, the Chu-Vandermonde formula (1.2) can be written as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(a)_k}{(b)_k} = \frac{(b-a)_n}{(b)_n},$$

and therefore we can conclude that the binomial transform of the sequence $\left\{ \frac{(a)_n}{(b)_n} \right\}_{n=0}^{\infty}$ is the sequence $\left\{ \frac{(b-a)_n}{(b)_n} \right\}_{n=0}^{\infty}$.

Theorem 2.2. *Let $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. Let $\{\tilde{x}_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ be defined by (2.1). Then for each $n \geq 0$, we have*

$$x_n = \sum_{k=0}^n (-1)^k \frac{(2k+1) \binom{n}{k}}{(n+k+1) \binom{n+k}{k}} \tilde{x}_k. \quad (2.7)$$

Equation (2.7) gives us the inverse transformation of the transformation defined in (2.1).

Using the formulas (2.2) and (2.3), we can also write equation (2.7) as

$$x_n = \sum_{k=0}^n \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} \tilde{x}_k, \quad n \geq 0. \quad (2.8)$$

Before we prove Theorem 2.2, we need the following lemma:

Lemma 2.3. *For each $n \geq 1$, we have*

$$\sum_{k=0}^{n-1} (-1)^k (2k+1) \binom{2n+1}{n-k} = (-1)^{n-1} (2n+1). \quad (2.9)$$

Proof. The result is directly verified when $n = 1, 2, 3, 4$. Assume $n \geq 5$. We let

$$A = \sum_{k=0}^{n-1} (-1)^k (2k+1) \binom{2n+1}{n-k}.$$

We split off the first two terms and the last two terms of A to get

$$\begin{aligned} A &= \binom{2n+1}{n} - 3\binom{2n+1}{n-1} + \sum_{k=2}^{n-3} (-1)^k (2k+1) \binom{2n+1}{n-k} \\ &\quad + (-1)^{n-2} (2n-3) \binom{2n+1}{2} + (-1)^{n-1} (2n-1) \binom{2n+1}{1}. \end{aligned}$$

For $0 \leq k \leq n-2$, we apply (1.3) twice to $\binom{2n+1}{n-k}$ and obtain

$$\begin{aligned} \binom{2n+1}{n-k} &= \binom{2n}{n-k} + \binom{2n}{n-k-1} \\ &= \binom{2n-1}{n-k} + 2\binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2}. \end{aligned}$$

Also, by (1.3),

$$\binom{2n+1}{1} = \binom{2n}{1} + \binom{2n}{0} = \binom{2n-1}{1} + \binom{2n-1}{0} + \binom{2n}{0}.$$

Therefore,

$$\begin{aligned} A &= \binom{2n-1}{n} + 2\binom{2n-1}{n-1} + \binom{2n-1}{n-2} \\ &\quad - 3\left(\binom{2n-1}{n-1} + 2\binom{2n-1}{n-2} + \binom{2n-1}{n-3}\right) \\ &\quad + \sum_{k=2}^{n-3} (-1)^k (2k+1) \left(\binom{2n-1}{n-k} + 2\binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2}\right) \\ &\quad + (-1)^{n-2} (2n-3) \left(\binom{2n-1}{2} + 2\binom{2n-1}{1} + \binom{2n-1}{0}\right) \\ &\quad + (-1)^{n-1} (2n-1) \left(\binom{2n-1}{1} + \binom{2n-1}{0} + \binom{2n}{0}\right). \end{aligned}$$

From here, we can write

$$A = A_1 + A_2 + A_3,$$

where

$$A_1 = \binom{2n-1}{n} + 2\binom{2n-1}{n-1} - 3\binom{2n-1}{n-1},$$

$$\begin{aligned}
A_2 &= \binom{2n-1}{n-2} - 3 \left(2 \binom{2n-1}{n-2} + \binom{2n-1}{n-3} \right) \\
&+ \sum_{k=2}^{n-3} (-1)^k (2k+1) \left(\binom{2n-1}{n-k} + 2 \binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2} \right) \\
&+ (-1)^{n-2} (2n-3) \left(\binom{2n-1}{2} + 2 \binom{2n-1}{1} \right) \\
&+ (-1)^{n-1} (2n-1) \binom{2n-1}{1},
\end{aligned}$$

and

$$A_3 = (-1)^{n-2} (2n-3) \binom{2n-1}{0} + (-1)^{n-1} (2n-1) \left(\binom{2n-1}{0} + \binom{2n}{0} \right).$$

Since $\binom{2n-1}{n} = \binom{2n-1}{n-1}$, we have that $A_1 = 0$.

To evaluate A_2 , we have

$$\begin{aligned}
A_2 &= \sum_{k=0}^{n-3} (-1)^k (2k+1) \binom{2n-1}{n-k-2} \\
&+ \sum_{k=1}^{n-2} (-1)^k (2k+1) (2) \binom{2n-1}{n-k-1} + \sum_{k=2}^{n-1} (-1)^k (2k+1) \binom{2n-1}{n-k} \\
&= \sum_{k=1}^{n-2} (-1)^{k-1} (2k-1) \binom{2n-1}{n-k-1} + \sum_{k=1}^{n-2} (-1)^k (4k+2) \binom{2n-1}{n-k-1} \\
&+ \sum_{k=1}^{n-2} (-1)^{k+1} (2k+3) \binom{2n-1}{n-k-1} = 0,
\end{aligned}$$

where the last equality follows by combining the three sums into one and factoring out $(-1)^{k-1} \binom{2n-1}{n-k-1}$.

Finally,

$$A_3 = (-1)^{n-2} (2n-3) + (-1)^{n-1} (4n-2) = (-1)^{n-1} (2n+1).$$

Therefore,

$$A = A_1 + A_2 + A_3 = (-1)^{n-1} (2n+1),$$

which completes the proof. \square

Proof of Theorem 2.2. We will prove that equation (2.8) holds for each $n \geq 0$. In our proof, we will use the equivalent form (2.4) of (2.1).

For $n, m \geq 0$, we define

$$a_{n,m} = \frac{(2m+1)(-n)_m}{(n+1)_{m+1}}.$$

We note that $a_{n,m} = 0$ for $m > n$, since $(-n)_m = 0$ for $m > n$. In order to prove the theorem, we need to show that for each $m \geq 0$, the transformation (2.4) of the sequence $\{a_{n,m}\}_{n=0}^{\infty}$ is given by

$$\tilde{a}_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \quad (2.10)$$

When $m = 0$, we have $\{a_{n,0}\}_{n=0}^{\infty} = \{\frac{1}{n+1}\}_{n=0}^{\infty}$. Using the Chu-Vandermonde formula (1.2), we have

$$\begin{aligned} \tilde{a}_{n,0} &= \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{k!k!} \frac{1}{k+1} \\ &= \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{k!(2)_k} = {}_2F_1 \left[\begin{matrix} -n, n+1; \\ 2; \end{matrix} 1 \right] \\ &= \frac{(1-n)_n}{(2)_n} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \end{aligned}$$

Therefore (2.10) holds for $m = 0$. Assume now that (2.10) holds for all $m = 0, 1, \dots, p$, for some $p \geq 0$. We will show that (2.10) holds for $m = p+1$ and the result will follow by induction.

For $n \geq 0$, we define

$$b_{n,p+1} = \frac{1}{(n+1)_{p+2}}.$$

Using the Chu-Vandermonde formula (1.2), we have

$$\begin{aligned} \tilde{b}_{n,p+1} &= \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{k!k!} \frac{1}{(k+1)_{p+2}} \\ &= \frac{1}{(p+2)!} \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{k!(p+3)_k} = \frac{1}{(p+2)!} {}_2F_1 \left[\begin{matrix} -n, n+1; \\ p+3; \end{matrix} 1 \right] \\ &= \frac{1}{(p+2)!} \frac{(p+2-n)_n}{(p+3)_n} = \frac{(p+2-n)_n}{(p+n+2)!}. \end{aligned}$$

When $n > p + 1$, we have that $(p + 2 - n)_n = 0$ and so $\tilde{b}_{n,p+1} = 0$. When $n = p + 1$, we have

$$\tilde{b}_{p+1,p+1} = \frac{(1)_{p+1}}{(2p+3)!} = \frac{1}{(p+2)_{p+2}}.$$

Using the induction hypothesis and the fact that the transformation (2.4) is linear, it follows that if we define

$$c_{n,p+1} = (p+2)_{p+2}b_{n,p+1} - \sum_{m=0}^p \frac{(p+2)_{p+2}(p+2-m)_m}{(p+m+2)!} a_{n,m}, \quad n \geq 0,$$

then we will have

$$\tilde{c}_{n,p+1} = \begin{cases} 1, & n = p+1, \\ 0, & n \neq p+1. \end{cases}$$

It remains to show that $c_{n,p+1} = a_{n,p+1}$ for all $n \geq 0$.

We compute

$$\frac{(p+2)_{p+2}(p+2-m)_m}{(p+m+2)!} = \frac{(2p+3)!}{(p+1-m)!(p+m+2)!} = \binom{2(p+1)+1}{p+1-m}.$$

Hence for each $n \geq 0$,

$$c_{n,p+1} = \frac{(p+2)_{p+2}}{(n+1)_{p+2}} - \sum_{m=0}^p \binom{2(p+1)+1}{p+1-m} \frac{(2m+1)(-n)_m}{(n+1)_{m+1}}. \quad (2.11)$$

By combining all terms under a common denominator, it follows that we can write $c_{n,p+1} = \frac{f(n)}{(n+1)_{p+2}}$, where $f(n)$ is a polynomial in n of degree at most $p+1$. Now since $\tilde{c}_{n,p+1} = 0$ for $n = 0, 1, \dots, p$, we must have $c_{n,p+1} = 0$ for $n = 0, 1, \dots, p$. But then $f(n) = 0$ for $n = 0, 1, \dots, p$ and so we must have $f(n) = \alpha \prod_{i=0}^p (n-i)$ for some $\alpha \in \mathbb{R}$. In view of (2.11),

$$\alpha = - \sum_{m=0}^p (-1)^m (2m+1) \binom{2(p+1)+1}{p+1-m}.$$

By Lemma 2.3, $\alpha = (-1)^{p+1}(2(p+1)+1)$. Therefore,

$$\begin{aligned} c_{n,p+1} &= \frac{(-1)^{p+1}(2(p+1)+1) \prod_{i=0}^p (n-i)}{(n+1)_{p+2}} \\ &= \frac{(2(p+1)+1)(-n)_{p+1}}{(n+1)_{p+2}} = a_{n,p+1} \text{ for all } n \geq 0, \end{aligned}$$

which shows that (2.10) holds for $m = p + 1$ and completes the proof by induction. \square

Interesting consequences of Theorem 2.2 are given below:

Corollary 2.4. *We have the following identities:*

(a) For $0 \leq m \leq n$,

$$\sum_{k=m}^n (-1)^k \frac{(2k+1) \binom{n-m}{k-m}}{(n+k+1) \binom{n+k}{m+k}} = \begin{cases} (-1)^n, & m = n, \\ 0, & 0 \leq m < n. \end{cases} \quad (2.12)$$

(b) For $n \geq 0$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{1}{2k+1} = \frac{1}{2n+1}. \quad (2.13)$$

Proof. (a) Using Theorem 2.2 and then switching the order of summation, we have that for every $n \geq 0$,

$$\begin{aligned} x_n &= \sum_{k=0}^n (-1)^k \frac{(2k+1) \binom{n}{k}}{(n+k+1) \binom{n+k}{k}} \tilde{x}_k \\ &= \sum_{k=0}^n \left((-1)^k \frac{(2k+1) \binom{n}{k}}{(n+k+1) \binom{n+k}{k}} \left(\sum_{m=0}^k (-1)^m \binom{k}{m} \binom{k+m}{m} x_m \right) \right) \\ &= \sum_{m=0}^n \left((-1)^m \left(\sum_{k=m}^n (-1)^k \frac{(2k+1) \binom{n}{k} \binom{k}{m} \binom{k+m}{m}}{(n+k+1) \binom{n+k}{k}} \right) x_m \right) \\ &= \sum_{m=0}^n \left((-1)^m \binom{n}{m}^2 \left(\sum_{k=m}^n (-1)^k \frac{(2k+1) \binom{n-m}{k-m}}{(n+k+1) \binom{n+k}{m+k}} \right) x_m \right), \end{aligned}$$

where the last equality follows from (1.4) and (1.5). Hence we must have

$$(-1)^m \binom{n}{m}^2 \left(\sum_{k=m}^n (-1)^k \frac{(2k+1) \binom{n-m}{k-m}}{(n+k+1) \binom{n+k}{m+k}} \right) = \begin{cases} 1, & m = n, \\ 0, & 0 \leq m < n, \end{cases}$$

and this implies (2.12).

(b) It is enough to show that the sequence $\{\frac{1}{2n+1}\}_{n=0}^{\infty}$ is fixed by the inverse transformation of (2.1). Indeed, we have

$$\begin{aligned} \sum_{k=0}^n \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} \frac{1}{2k+1} &= \sum_{k=0}^n \frac{(-n)_k}{(n+1)_{k+1}} = \frac{1}{n+1} \sum_{k=0}^n \frac{(-n)_k}{(n+2)_k} \\ \frac{1}{n+1} {}_2F_1 \left[\begin{matrix} -n, 1; \\ n+2; \end{matrix} 1 \right] &= \frac{1}{n+1} \frac{(n+1)_n}{(n+2)_n} = \frac{1}{2n+1}, \end{aligned}$$

where in the next-to-last step we used the Chu-Vandermonde formula (1.2). \square

The special case $m = 0$ in (2.12) gives

$$\sum_{k=0}^n (-1)^k \frac{(2k+1) \binom{n}{k}}{(n+k+1) \binom{n+k}{k}} = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \quad (2.14)$$

In view of (2.8), we can also write (2.14) as

$$\sum_{k=0}^n \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \quad (2.15)$$

We note that (2.13) implies that the sequence $\{\frac{1}{2n+1}\}_{n=0}^{\infty}$ is fixed by the transformation (2.1). In fact, since in the last term of the sum

$$\tilde{x}_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} x_k$$

the coefficient in front of x_n is $(-1)^n \binom{2n}{n} \neq 1$ for $n > 0$, it follows that, up to constant multiples, the sequence $\{\frac{1}{2n+1}\}_{n=0}^{\infty}$ is the only one fixed by the transformation (2.1).

Corollary 2.5. *Let $0 \leq s \leq T - 1$. Then for every m such that $0 \leq m \leq T - 1$, we have*

$$\sum_{n=0}^m (-1)^n \frac{(2n+1) \binom{m}{n}}{(m+n+1) \binom{m+n}{n}} R_n(s, T) = \frac{(-s)_m (s+1)_m}{(1-T)_m (1+T)_m}. \quad (2.16)$$

Proof. Let

$$x_n = \begin{cases} \frac{(-s)_n(s+1)_n}{(1-T)_n(1+T)_n}, & 0 \leq n \leq T-1 \\ 0, & n > T-1. \end{cases}$$

Then for $0 \leq n \leq T-1$,

$$\begin{aligned} R_n(s, T) &= {}_4F_3 \left[\begin{matrix} -n, n+1, -s, s+1; \\ 1, 1-T, 1+T; \end{matrix} 1 \right] \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{(-s)_k(s+1)_k}{(1-T)_k(1+T)_k} = \tilde{x}_n \end{aligned}$$

Theorem 2.2 now yields the result. \square

In view of (2.8), we can also write (2.16) as

$$\sum_{n=0}^m \frac{(2n+1)(-m)_n}{(m+1)_{n+1}} R_n(s, T) = \frac{(-s)_m(s+1)_m}{(1-T)_m(1+T)_m}, \quad (2.17)$$

for all m such that $0 \leq m \leq T-1$.

We note that $\frac{(-s)_m(s+1)_m}{(1-T)_m(1+T)_m} = 0$ if $s+1 \leq m \leq T-1$, and so we have

$$\sum_{n=0}^m (-1)^n \frac{(2n+1) \binom{m}{n}}{(m+n+1) \binom{m+n}{n}} R_n(s, T) = 0, \quad s+1 \leq m \leq T-1, \quad (2.18)$$

or, equivalently,

$$\sum_{n=0}^m \frac{(2n+1)(-m)_n}{(m+1)_{n+1}} R_n(s, T) = 0, \quad s+1 \leq m \leq T-1. \quad (2.19)$$

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